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On a singular initial-value problem for the Navier-Stokes equations

L. E. Fraenkel and M. D. Preston

This paper presents a recent result for the problem introduced eleven years ago in [1], but described only briefly there. We shall prove the following, as far as space allows. The vorticity ω of a diffusing vortex circle in a viscous fluid has, for small values of a non-dimensional time, a second approximation $\omega_A + \omega_1$ that, although formulated for a fixed, finite Reynolds number λ and exact for $\lambda = 0$ (then $\omega = \omega_A$), tends to a smooth limiting function as $\lambda \uparrow \infty$.

In §1 and §2 the necessary background and apparatus are described; §3 outlines the new result and its proof.

1 Introduction

In a certain weak sense, this paper is a continuation of [1]. However, no knowledge of [1] is required if the reader is willing to accept that a vorticity field in \mathbb{R}^3 (subject to mild restrictions, but not required to have any symmetry) has a *centroid of vorticity* moving with a velocity $\mathbf{U}(t)$ that is given by an explicit formula when the vorticity $\omega(\cdot, t)$ throughout \mathbb{R}^3 is known. This result is essentially due to Saffman [2]; it was generalized a little (and perhaps clarified and sharpened) in [1].

We seek a solution of the Navier-Stokes equations with the initial condition illustrated in Figure 1: at time zero, vorticity ω is concentrated on, and is tangential to, a horizontal circle in \mathbb{R}^3 . This initial vorticity induces an initial velocity field that has infinite kinetic energy. (The circle then diffuses and moves vertically, at first with infinite velocity; at all positive times $t > 0$ the kinetic energy is finite.)

More precisely, consider incompressible fluid occupying all of \mathbb{R}^3 and at rest at infinity there; let $x := (x_1, x_2, x_3)$ be such that the frame (Ox_1, Ox_2, Ox_3) moves, relative to the motionless fluid at infinity, with the velocity $(0, 0, U(t))$ of the centroid of vorticity, the axes remaining parallel to their initial positions. The *fluid velocity* relative to this moving frame is written $\mathbf{v}(x, t)$ and the *vorticity* is

$$\omega := \text{curl } \mathbf{v} = \nabla \times \mathbf{v}.$$

Our *time variable* is $t = \nu T$, where T denotes physical time and ν is the kinematic viscosity (a given positive constant). This choice of t simplifies the heat operator in (1.3) below and simplifies most subsequent equations.

In writing $\mathbf{U}(t) = (0, 0, U(t))$, we have restricted attention to the cylindrical symmetry implied by the initial condition

$$\omega(x, 0) = \kappa \delta(z) \delta(r - a) \mathbf{e}^\phi, \tag{1.1}$$

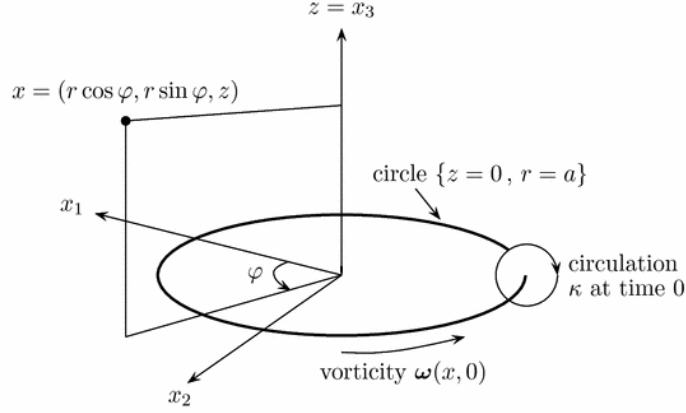


Figure 1: The initial condition.

in which the circulation κ and the radius a are given positive constants, cylindrical co-ordinates (z, r, ϕ) are defined by $x =: (r \cos \phi, r \sin \phi, z)$, the unit vector $\mathbf{e}^\phi := (-\sin \phi, \cos \phi, 0)$ and δ denotes the Dirac generalized function.

In terms of the vorticity $\boldsymbol{\omega}$, the fluid velocity (relative to our moving frame) is

$$\mathbf{v}(x, t) = -(0, 0, U(t)) + \nabla \times \int_{\mathbb{R}^3} \frac{1}{4\pi|x - x'|} \boldsymbol{\omega}(x', t) dx'. \quad (1.2)$$

With (1.1) and (1.2) understood, we seek $\boldsymbol{\omega}(x, t)$ such that

$$\left(\frac{\partial}{\partial t} - \Delta \right) \boldsymbol{\omega} = -\frac{1}{\nu} ((\mathbf{v} \cdot \nabla) \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla) \mathbf{v}) \quad \text{in } \mathbb{R}^3 \times (0, \bar{t}) \quad (1.3)$$

for some small $\bar{t} > 0$.

Of course, it would be better to solve the problem (1.1) to (1.3) for all $t > 0$, but this is beyond us because we seek rather explicit answers. There are two excuses for considering only small t , or, rather, small t/a^2 , which is non-dimensional. First, once a solution for $t > 0$ has been established, the general theory of the Navier-Stokes equations implies a continuation of the solution to all time, thanks to finite energy for $t > 0$, cylindrical symmetry and absence of a swirl velocity (of a velocity component in the direction \mathbf{e}^ϕ). Secondly, if the viscosity ν is small, which may be the case of primary interest, then the requirement that $\nu T/a^2$ be small does not demand that the physical time T be small.

In view of (1.1), we write

$$\boldsymbol{\omega}(x, t) =: \omega(z, r, t) \mathbf{e}^\phi,$$

and seek the solution of (1.1) to (1.3) in the scalar form $\omega = \omega_A + \omega_1 + \rho$, where ω_A is to be a first approximation for small t/a^2 and $\omega_A + \omega_1$ is to be a second (improved) approximation; the remainder ρ is to make $\omega_A + \omega_1 + \rho$ an exact solution and is to be $o(\omega_1)$ as $t \downarrow 0$. Here are some details.

(i) The non-linear terms on the right-hand side of (1.3) are expected to be small for small t/a^2 , because ω should be approximately constant and large on small circles in a meridional plane ($\phi = \text{constant}$) centred at $(z, r) = (0, a)$, so that \mathbf{v} is approximately tangential to such circles and approximately of constant magnitude on each of them. (If the initial vortex circle were a straight line, then these non-linear terms would vanish.) If the right-hand member of (1.3) is neglected, there results the formal approximation

$$\omega_A(z, r, t) = \frac{\kappa}{4\pi t} \exp\left(-\frac{s^2}{4t}\right) \left(\frac{a}{r}\right)^{1/2} B\left(\frac{ar}{2t}\right), \quad (1.4)$$

where $s := \{z^2 + (r - a)^2\}^{1/2}$ and B is a known function such that $B(y) \rightarrow 1$ as $y \rightarrow \infty$; in fact,

$$B(y) := (2\pi y)^{1/2} e^{-y} I_1(y) \quad (0 \leq y < \infty), \quad (1.5)$$

where I_1 is the modified Bessel function of the first kind and of order 1 (as in [3], p.77).

(ii) The exponential in (1.4) prompts us to introduce *inner variables*

$$\sigma := \frac{s}{(4t)^{1/2}}, \quad \theta := \tan^{-1} \frac{r - a}{z}; \quad (1.6)$$

then the amplitude $\kappa/4\pi t$ in (1.4) prompts us to pose

$$\omega_1(z, r, t) = (4t)^{-1/2} \tilde{\omega}_1(\sigma, \theta). \quad (1.7)$$

It suffices to consider ω_1 in an *inner region*: $t \downarrow 0$ with σ fixed, so that $s \downarrow 0$, because in an *outer region*: $t \downarrow 0$ with $s \geq \text{constant} > 0$, not only ω_A , but also ω , are exponentially small.

The rest of this paper is devoted mainly to description of $\tilde{\omega}_1$; the *Reynolds number*

$$\lambda := \frac{\kappa}{2\pi\nu} \quad (1.8)$$

will be an important parameter.

(iii) The problem for the remainder ρ was sketched in [1]; the function $\rho(z, r, t)$ must be shown to exist and to be suitably small on the whole set $\mathbb{R} \times [0, \infty) \times (0, \bar{t}]$. Considerable progress has been made with this problem since [1] was written, but this analysis (which can only estimate ρ) is too long and too elaborate to be described here.

2 The perturbation ω_1 for fixed Reynolds number λ

With ω_1 as in (1.7), we adopt the notation

$$\begin{aligned} \text{(a)} \quad & (\sigma, \theta) \in E := (0, \infty) \times (-\pi, \pi], \\ \text{(b)} \quad & \Delta_\sigma := \left(\frac{\partial}{\partial \sigma}\right)^2 + \frac{1}{\sigma} \frac{\partial}{\partial \sigma} + \frac{1}{\sigma^2} \left(\frac{\partial}{\partial \theta}\right)^2, \\ \text{(c)} \quad & (A\tilde{\omega}_1)(\sigma_0, \theta_0) := \frac{1}{2\pi} \iint_E \log \frac{1}{|\sigma e^{i\theta} - \sigma_0 e^{i\theta_0}|} \tilde{\omega}_1(\sigma, \theta) \sigma d\sigma d\theta, \\ \text{(d)} \quad & \omega_{A,0}(\sigma, t) := \frac{\kappa}{4\pi t} e^{-\sigma^2}, \end{aligned} \quad (2.1)$$

in which $A\tilde{\omega}_1$ is a stream function describing the plane flow induced by vorticity $\tilde{\omega}_1$; the approximation $\omega_{A,0}$ to ω_A is that appropriate to $t \downarrow 0$ with σ fixed. We seek $\tilde{\omega}_1(\sigma, \theta)$ by linearizing (1.2) and (1.3) about $\omega_{A,0}$; the problem is then to solve the equation

$$\begin{aligned} -\left(\Delta_\sigma + 2\sigma \frac{\partial}{\partial \sigma} + 2\right)\tilde{\omega}_1 + \lambda \frac{1 - e^{-\sigma^2}}{\sigma^2} \frac{\partial}{\partial \theta} \tilde{\omega}_1 - 4\lambda e^{-\sigma^2} \frac{\partial}{\partial \theta} (A\tilde{\omega}_1) \\ = \frac{\kappa\lambda}{\pi a} g(\sigma) \cos \theta \quad \text{on } E, \end{aligned} \quad (2.2)$$

with side conditions

$$\tilde{\omega}_1(\sigma, \theta) \rightarrow 0 \quad \text{as } \sigma \downarrow 0 \text{ and as } \sigma \uparrow \infty. \quad (2.3)$$

The function g is a known, smooth function such that

- (a) $g(\sigma) = O(\sigma)$ as $\sigma \downarrow 0$;
- (b) $g(\sigma) = O(\sigma \log \sigma e^{-\sigma^2})$ as $\sigma \uparrow \infty$;

in fact,

$$(c) \quad g(\sigma) := \sigma e^{-\sigma^2} \left(\frac{3}{2} \frac{1 - e^{-\sigma^2}}{\sigma^2} + \left(\log \frac{1}{\sigma} - \int_\sigma^\infty \frac{e^{-\rho^2}}{\rho} d\rho \right) - \frac{1}{2}(\gamma_E + 1 - \log 2) \right), \quad (2.4)$$

where $\gamma_E = 0.5772\dots$ denotes Euler's constant.

Theorem 2.1. *For fixed $\lambda \in [0, \infty)$, the problem (2.2) and (2.3) for $\tilde{\omega}_1$ has a pointwise, unique solution; in particular, $\tilde{\omega}_1(\cdot, \theta) \in C^\infty[0, \infty)$, $\tilde{\omega}_1(0, \theta) = 0$ and $\tilde{\omega}_1(\sigma, \theta) = o(e^{-\sigma^2/2})$ as $\sigma \uparrow \infty$.*

Here we have space only to sketch the main steps of the proof.

(i) Under the transformation

$$\tilde{\omega}_1(\sigma, \theta) = e^{-\sigma^2/2} q(\sigma, \theta), \quad (2.5)$$

equation (2.2) becomes

$$\begin{aligned} -(\Delta_\sigma - \sigma^2)q + \lambda \frac{1 - e^{-\sigma^2}}{\sigma^2} \frac{\partial q}{\partial \theta} - 4\lambda e^{-\sigma^2/2} T(e^{-\sigma^2/2} q) \\ = \lambda e^{\sigma^2/2} f(\sigma, \theta) \quad \text{on } E, \end{aligned} \quad (2.6)$$

where the operator $T := (\partial/\partial \theta)A$ and $f(\sigma, \theta) := (\kappa/\pi a)g(\sigma) \cos \theta$. Let

$$(\xi, \eta) := \sigma(\cos \theta, \sin \theta), \quad q_*(\xi, \eta) = q_*(\sigma \cos \theta, \sigma \sin \theta) := q(\sigma, \theta). \quad (2.7)$$

The condition in (2.3) for $\sigma \downarrow 0$ will be implicit in what follows; it was imposed only to make q_* decent at the origin, because we shall find that q is of form $q_c(\sigma) \cos \theta + q_s(\sigma) \sin \theta$. Henceforth the functions q_* and q will be identified wherever no confusion is possible. Similarly, the Cartesian-co-ordinate and polar-co-ordinate representations of other functions will be identified.

(ii) In the first instance we establish a weak solution of (2.6). Let the *real Hilbert space* Z be the completion of the set $C_c^\infty(\mathbb{R}^2)$, of real-valued, infinitely

differentiable functions on \mathbb{R}^2 having compact support, in the norm implied by the inner product

$$\langle u, v \rangle_Z := \iint_{\mathbb{R}^2} (\nabla u \cdot \nabla v + \sigma^2 uv) \, d\xi d\eta. \quad (2.8)$$

We shall say that q is a *weak solution* of (2.6) if (and only if) $q \in Z$ and, for all test functions $u \in Z$,

$$\begin{aligned} B(u, q) := \iint_{\mathbb{R}^2} \left(\nabla u \cdot \nabla q + \sigma^2 uq + \lambda \frac{1 - e^{-\sigma^2}}{\sigma^2} u \frac{\partial q}{\partial \theta} \right. \\ \left. - 4\lambda e^{-\sigma^2/2} u T(e^{-\sigma^2/2} q) \right) d\xi d\eta = \lambda \iint_{\mathbb{R}^2} e^{\sigma^2/2} f u \, d\xi d\eta. \end{aligned} \quad (2.9)$$

(iii) Here is the key step of the proof.

Lemma 2.2. *The bilinear form B satisfies, for all u and v in Z ,*

$$B(u, u) = \|u\|^2, \quad (2.10)$$

$$|B(u, v)| \leq (1 + k_B \lambda) \|u\| \|v\|, \quad (2.11)$$

where $\|\cdot\| = \|\cdot\|_Z$ and k_B is an absolute constant (independent of the variables, parameters and functions in question).

Partial proof. We shall prove only that (2.10) holds for all functions in $C_c^\infty(\mathbb{R}^2)$. The remainder of the proof is neither trivial nor immediate, but it is of a kind familiar in Sobolev-space theory and its application to partial differential equations.

In view of the definition of B in (2.9), we wish to prove that, for all $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\iint_{\mathbb{R}^2} \frac{1 - e^{-\sigma^2}}{\sigma^2} \varphi \frac{\partial \varphi}{\partial \theta} \, d\xi d\eta = 0$$

and

$$\iint_{\mathbb{R}^2} e^{-\sigma^2/2} \varphi T(e^{-\sigma^2/2} \varphi) \, d\xi d\eta = 0.$$

The first of these is immediate because $\int_{-\pi}^{\pi} \varphi \frac{\partial \varphi}{\partial \theta} \, d\theta = 0$. For the second, let $A(e^{-\sigma^2/2} \varphi) =: \psi$; then $e^{-\sigma^2/2} \varphi = -\Delta \psi$ and we wish to prove that

$$-\iint_{\mathbb{R}^2} (\Delta \psi) \frac{\partial \psi}{\partial \theta} \, d\xi d\eta = 0.$$

Here it suffices to integrate over an open disc (or ball) $\mathcal{B}(0, R)$ with centre the origin and radius R so large that $\mathcal{B}(0, R)$ contains the compact support of $\Delta \psi$. Thus the integral may be written

$$-\int_{\partial \mathcal{B}(0, R)} \frac{\partial \psi}{\partial \sigma} \frac{\partial \psi}{\partial \theta} R \, d\theta + \iint_{\mathcal{B}(0, R)} \nabla \psi \cdot \frac{\partial}{\partial \theta} \nabla \psi \, d\xi d\eta.$$

That this last integral over $\mathcal{B}(0, R)$ vanishes is immediate as before. The boundary integral is now independent of R and vanishes because $\partial \psi / \partial \sigma$ and

$\partial\psi/\partial\theta$ are both $O(R^{-1})$ as $R \uparrow \infty$, by the definition (2.1)(c) of the operator A .
 \square

(iv) Existence and uniqueness of a weak solution. The forcing function in (2.6) satisfies amply the condition

$$\iint_{\mathbb{R}^2} \frac{e^{\sigma^2} f(\sigma, \theta)^2}{1 + \sigma^2} d\xi d\eta < \infty, \quad (2.12)$$

because $f(\sigma, \theta) = (\kappa/\pi a)g(\sigma) \cos \theta$ with g as in (2.4). This condition is sufficient to make the forcing integral in (2.9), namely,

$$F(u) := \iint_{\mathbb{R}^2} e^{\sigma^2/2} f u d\xi d\eta, \quad u \in Z,$$

a bounded linear functional evaluated at u . In other words, F belongs to the dual space Z^* of Z . The Lax-Milgram lemma now implies

Lemma 2.3. *Equation (2.6) has a unique weak solution q and*

$$\frac{\lambda}{1 + k_B \lambda} \|F\|_{Z^*} \leq \|q\|_Z \leq \lambda \|F\|_{Z^*}. \quad (2.13)$$

(v) Regularity theory: pointwise estimates. We separate the variables σ and θ . Let Y denote the real Hilbert space of functions $y : [0, \infty) \rightarrow \mathbb{R}$ such that the functions having values $y(\sigma) \cos \theta$ or $y(\sigma) \sin \theta$ belong to Z . It can be proved that, equivalently, Y is the completion of the set

$$D := \{\zeta \in C_c^\infty[0, \infty) \mid \zeta(0) = 0\},$$

where the compact support of ζ may extend to the origin, in the norm implied by the inner product

$$\langle v, w \rangle_Y := \int_0^\infty \left(v' w' + \left(\frac{1}{\sigma^2} + \sigma^2 \right) v w \right) \sigma d\sigma,$$

where the $(\cdot)'$ denotes differentiation.

It can then be proved that, if

$$(a) \quad Q(\sigma) := q_c(\sigma) + i q_s(\sigma), \quad \text{where } (q_c, q_s) \in Y^2; \quad (2.14)$$

(b) the operator T_1 is defined by

$$(T_1 y)(\sigma) := \frac{1}{2} \int_0^\infty \left(\frac{\rho}{\sigma} \wedge \frac{\sigma}{\rho} \right) y(\rho) \rho d\rho \quad \text{for all } y \in Y, \quad (2.15)$$

where $a \wedge b$ denotes the lower envelope, or lesser, of a and b ;

(c) for all test functions $v \in Y$,

$$\begin{aligned} & \int_0^\infty \left(v' Q' + \left(\frac{1}{\sigma^2} + \sigma^2 \right) v Q - i \lambda \frac{1 - e^{-\sigma^2}}{\sigma^2} v Q + i 4 \lambda e^{-\sigma^2/2} v T_1(e^{-\sigma^2/2} Q) \right) \sigma d\sigma \\ &= \lambda \int_0^\infty e^{\sigma^2/2} f_c v \sigma d\sigma, \end{aligned} \quad (2.16)$$

where $f_c(\sigma) := (\kappa/\pi a)g(\sigma)$;
(d)

$$q(\sigma, \theta) := q_c(\sigma) \cos \theta + q_s(\sigma) \sin \theta; \quad (2.17)$$

then q satisfies (2.9), so that the right-hand member of (2.17) is the unique weak solution of (2.6). Conversely, equations (2.9), (2.17) and (2.14) imply (2.16).

We now choose the test function in (2.16) to be a Green function of the operator

$$-\left(\frac{d}{d\sigma}\right)^2 - \frac{1}{\sigma} \frac{d}{d\sigma} + \left(\frac{1}{\sigma^2} + \sigma^2\right),$$

which results from insertion of (2.17) into (2.6). It is legitimate to choose

$$v(\sigma) = K(\rho, \sigma) := \begin{cases} \frac{1}{\sigma} \sinh \frac{\sigma^2}{2} \cdot \frac{1}{\rho} \exp\left(-\frac{\rho^2}{2}\right) & \text{if } \sigma \leq \rho, \\ \frac{1}{\sigma} \exp\left(-\frac{\sigma^2}{2}\right) \cdot \frac{1}{\rho} \sinh \frac{\rho^2}{2} & \text{if } \sigma \geq \rho, \end{cases} \quad (2.18)$$

because $K(\rho, \cdot) \in Y$ for fixed $\rho \in (0, \infty)$. Then (2.16) yields, after an integration by parts, the integral equation

$$\begin{aligned} Q(\rho) = & \lambda \int_0^\infty K(\rho, \sigma) e^{\sigma^2/2} f_c(\sigma) \sigma d\sigma \\ & + i\lambda \int_0^\infty K(\rho, \sigma) \left(\frac{1 - e^{-\sigma^2}}{\sigma^2} Q(\sigma) - 4e^{-\sigma^2/2} T_1(e^{-\sigma^2/2} Q) \right) \sigma d\sigma. \end{aligned} \quad (2.19)$$

Since Lemma 2.3 provides bounds for $\|q_c\|_Y$ and $\|q_s\|_Y$, the regularity of Q , and pointwise bounds, can be deduced from (2.19) and from Lemma 3.8 below without great difficulty.

3 The perturbation ω_1 as $\lambda \uparrow \infty$

We return to equations (2.1) to (2.4) and define a stream function $\tilde{\psi}_1 := A\tilde{\omega}_1$. Then $\tilde{\omega}_1 = -\Delta_\sigma \tilde{\psi}_1$ and (2.2) becomes

$$\begin{aligned} \left(\Delta_\sigma + 2\sigma \frac{\partial}{\partial \sigma} + 2 \right) \Delta_\sigma \tilde{\psi}_1 - \lambda \frac{1 - e^{-\sigma^2}}{\sigma^2} \frac{\partial}{\partial \theta} \Delta_\sigma \tilde{\psi}_1 - 4\lambda e^{-\sigma^2} \frac{\partial \tilde{\psi}_1}{\partial \theta} \\ = \frac{\kappa \lambda}{\pi a} g(\sigma) \cos \theta \quad \text{on } E. \end{aligned} \quad (3.1)$$

In view of (2.5) and (2.17), the function $\tilde{\psi}_1$ has the form

$$\tilde{\psi}_1(\sigma, \theta) = \tilde{\psi}_{1c}(\sigma) \cos \theta + \tilde{\psi}_{1s}(\sigma) \sin \theta. \quad (3.2)$$

We divide (3.1) by $\lambda(1 - e^{-\sigma^2})/\sigma^2$, write the $\cos \theta$ and $\sin \theta$ parts as separate equations and define, similarly to (2.14),

$$\Psi(\sigma) = \tilde{\psi}_{1c}(\sigma) + i\tilde{\psi}_{1s}(\sigma). \quad (3.3)$$

With the notation

$$\begin{aligned}
\text{(a)} \quad \Delta_1 &:= \left(\frac{d}{d\sigma} \right)^2 + \frac{1}{\sigma} \frac{d}{d\sigma} - \frac{1}{\sigma^2}, \\
\text{(b)} \quad \alpha(\sigma) &:= \frac{4\sigma^2}{e^{\sigma^2} - 1}, \\
\text{(c)} \quad \beta(\sigma) &:= \frac{\sigma^2}{1 - e^{-\sigma^2}}, \\
\text{(d)} \quad \mathcal{E} &:= \Delta_1 + 2\sigma \frac{d}{d\sigma} + 2,
\end{aligned} \tag{3.4}$$

the problem for $\tilde{\omega}_1$ is to solve the equation

$$-i \frac{\beta(\sigma)}{\lambda} \mathcal{E}(\Delta_1 \Psi) + \{\Delta_1 + \alpha(\sigma)\} \Psi = -i \frac{\kappa}{\pi a} \beta(\sigma) g(\sigma), \quad 0 < \sigma < \infty, \tag{3.5}$$

with the side conditions

$$\begin{aligned}
&\text{as } \sigma \downarrow 0, \quad (\Delta_1 \Psi)(\sigma) \rightarrow 0, \quad \Psi'(\sigma) = O(1) \quad \text{and} \quad \Psi(\sigma) = O(\sigma); \\
&\text{as } \sigma \uparrow \infty, \quad (\Delta_1 \Psi)(\sigma) \rightarrow 0, \quad \Psi'(\sigma) = O(\sigma^{-2}) \quad \text{and} \quad \Psi(\sigma) = O(\sigma^{-1}).
\end{aligned} \tag{3.6}$$

Here the conditions on $\Delta_1 \Psi$ come from (2.3); the conditions on Ψ' and Ψ are implied by $\Psi = -T_1(\Delta_1 \Psi)$, with T_1 as in (2.15), and by conditions on $\Delta_1 \Psi$ much weaker than those in Theorem 2.1.

For $\lambda \uparrow \infty$, equation (3.5) with (3.6) seems to form a singular perturbation problem, since a small parameter multiplies the highest derivatives. Surprisingly, this turns out not to be the case; nevertheless there is work to be done.

Apparently, if $\Psi_0(\sigma) := \lim_{\lambda \uparrow \infty} \Psi(\sigma; \lambda)$ exists, then it must satisfy

$$\{\Delta_1 + \alpha(\sigma)\} \Psi_0 = -i \frac{\kappa}{\pi a} \beta(\sigma) g(\sigma), \quad 0 < \sigma < \infty, \tag{3.7}$$

and the six side conditions (3.6). We proceed to explore this problem.

Lemma 3.1. *The equation*

$$\{\Delta_1 + \alpha(\sigma)\} u = 0, \quad 0 < \sigma < \infty, \tag{3.8}$$

has solutions

$$U(\sigma) := \frac{1}{\sigma} \left(1 - e^{-\sigma^2} \right) \tag{3.9}$$

and

$$V(\sigma) := \frac{1}{\sigma} - U(\sigma) \log \left(e^{\sigma^2} - 1 \right). \tag{3.10}$$

Here U is an eigensolution (with eigenvalue 0) in that it satisfies not only (3.8) but also all six side conditions (3.6).

Proof. This is a matter of direct calculation. \square

Lemma 3.2. *The forcing function in (3.7) is orthogonal to the eigensolution U in the sense that*

$$\int_0^\infty U(\sigma) \beta(\sigma) g(\sigma) \sigma d\sigma = 0. \tag{3.11}$$

Hence the problem (3.7) and (3.6) has a (non-unique) solution

$$\Psi_0(\sigma) = c_0 U(\sigma) + i \frac{\kappa}{2\pi a} \int_0^\sigma \{U(\rho)V(\sigma) - U(\sigma)V(\rho)\} \beta(\rho)g(\rho) \rho d\rho \quad (3.12)$$

for every $c_0 \in \mathbb{C}$.

Proof. Again this is a matter of direct calculation, but the calculation is not short. With β defined by (3.4)(c) and g by (2.4)(c), the analytic proof of the orthogonality condition (3.11) seems to require a page. (However, with any machine capable of numerical integration, numerical verification of (3.11) is quick and easy.) We note that Liouville's formula for Wronskians yields

$$U(\sigma)V'(\sigma) - U'(\sigma)v(\sigma) = -\frac{2}{\sigma}, \quad 0 < \sigma < \infty. \quad (3.13)$$

The following lemma is also relevant. □

Lemma 3.3. *Define, for suitable functions f ,*

$$(\mathcal{G}f)(\sigma) := \frac{1}{2} \int_0^\sigma \{U(\rho)V(\sigma) - U(\sigma)V(\rho)\} f(\rho) \rho d\rho, \quad 0 < \sigma < \infty, \quad (3.14)$$

and

$$J(f) := \int_0^\infty V(\rho)f(\rho) \rho d\rho. \quad (3.15)$$

Assume that $f \in C[0, \infty)$, that $\int_0^\infty U(\rho)f(\rho) \rho d\rho = 0$, that $f(\sigma) = O(\sigma)$ as $\sigma \downarrow 0$ and that $f(\sigma) = O(\sigma^m e^{-\sigma^2})$, with $m \geq 1$, as $\sigma \uparrow \infty$. Then

$$\{\Delta_1 + \alpha(\sigma)\}(\mathcal{G}f)(\sigma) = -f(\sigma) \quad \text{in } (0, \infty); \quad (3.16)$$

as $\sigma \downarrow 0$,

$$(\mathcal{G}f)(\sigma) = O(\sigma^3) \quad \text{and} \quad (\Delta_1 \mathcal{G}f)(\sigma) = O(\sigma); \quad (3.17)$$

as $\sigma \uparrow \infty$,

$$(\mathcal{G}f)(\sigma) = -\frac{1}{2}J(f)\sigma^{-1} + O(\sigma^m e^{-\sigma^2}), \quad (3.18)$$

$$(\mathcal{G}f)'(\sigma) = \frac{1}{2}J(f)\sigma^{-2} + O(\sigma^{m-1} e^{-\sigma^2}), \quad (3.19)$$

$$(\mathcal{G}f)''(\sigma) = -J(f)\sigma^{-3} + O(\sigma^m e^{-\sigma^2}), \quad (3.20)$$

and

$$(\Delta_1 \mathcal{G}f)(\sigma) \quad \text{and} \quad (\mathcal{E} \mathcal{G}f)(\sigma) \quad \text{are} \quad O(\sigma^m e^{-\sigma^2}). \quad (3.21)$$

Proof. Equation (3.16) follows from the definition of $\mathcal{G}f$ and two differentiations. That $(\mathcal{G}f)(\sigma) = O(\sigma^3)$ as $\sigma \downarrow 0$ also follows from the definition; then the differential equation (3.16) shows that $(\Delta_1 \mathcal{G}f)(\sigma) = O(\sigma)$ as $\sigma \downarrow 0$. In order to prove (3.18) and (3.19), we note that $U(\sigma) \sim 1/\sigma$ and $V(\sigma) \sim -\sigma$ as $\sigma \uparrow \infty$, whence

$$\begin{aligned} \int_0^\sigma U(\rho)f(\rho) \rho d\rho &= \left(\int_0^\infty - \int_\sigma^\infty \right) U(\rho)f(\rho) \rho d\rho = 0 + O(\sigma^{m-1} e^{-\sigma^2}), \\ \int_0^\sigma V(\rho)f(\rho) \rho d\rho &= \left(\int_0^\infty - \int_\sigma^\infty \right) V(\rho)f(\rho) \rho d\rho = J(f) + O(\sigma^{m+1} e^{-\sigma^2}), \end{aligned}$$

from which (3.18) and (3.19) follow.

The differential equation (3.16) and the estimate (3.18) of $\mathcal{G}f$ imply that $(\Delta_1 \mathcal{G}f)(\sigma) = O(\sigma^m e^{-\sigma^2})$ as $\sigma \uparrow \infty$. What has been proved now implies the estimates of $(\mathcal{G}f)''(\sigma)$ and $(\mathcal{E}\mathcal{G}f)(\sigma)$ for $\sigma \uparrow \infty$. \square

The result (3.12) prompts two questions. How (if at all) is c_0 to be evaluated? How smooth is Ψ_0 ? Analogues of both these questions will have to be answered more generally for each function Ψ_n in an identity

$$\Psi(\sigma; \lambda) = \sum_{n=0}^N \lambda^{-n} \Psi_n(\sigma) + R_N(\sigma; \lambda).$$

Here we anticipate later results and note that, with J as in (3.15),

$$c_0 = i \frac{\kappa}{2\pi a} J(\beta g) = i \frac{\kappa}{a} (0.11527\dots). \quad (3.22)$$

This follows from the equation governing Ψ_1 , which requires an orthogonality condition involving Ψ_0 .

Because of the function $Q =: Q(\cdot; \lambda)$ defined by (2.14) and (2.16), we now define $Q_0 = q_{0c} + iq_{0s}$ by

$$\begin{aligned} Q_0(\sigma) &:= -e^{\sigma^2/2} (\Delta_1 \Psi_0)(\sigma) \\ &= e^{\sigma^2/2} \left(4c_0 \sigma e^{-\sigma^2} - \alpha(\sigma) (\mathcal{G}h_0)(\sigma) - h_0(\sigma) \right), \end{aligned} \quad (3.23)$$

where $h_0 := -i(\kappa/\pi a)\beta g$. Evidently $q_{0c} = 0$.

It will emerge from Theorem 3.5 that Q_0 is the limit of $Q(\cdot; \lambda)$ as $\lambda \uparrow \infty$. In Figures 2 and 3, Q_0 is compared with $Q(\cdot; \lambda)$ for large λ ; these values of $Q(\cdot; \lambda)$ were obtained by numerical solution of the equation

$$-(\Delta_1 - \sigma^2)Q - i\lambda \frac{1 - e^{-\sigma^2}}{\sigma^2} Q + i4\lambda e^{-\sigma^2/2} T_1(e^{-\sigma^2/2} Q) = \frac{\kappa\lambda}{\pi a} e^{\sigma^2/2} g(\sigma). \quad (3.24)$$

This equation is equivalent to (2.6), because of (2.17); it is also the pointwise form of (2.16); with the condition that $Q(\sigma) \rightarrow 0$ as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$, it has a pointwise, unique solution. Figures 2 and 3 are consistent with the result of Theorem 3.5 that, as $\lambda \uparrow \infty$,

$$q_c(\cdot; \lambda) = O(\lambda^{-1}) \quad \text{and} \quad q_s(\cdot; \lambda) - q_{0s} = O(\lambda^{-2}).$$

Definition. We shall say that a function $\varphi : [0, \infty) \rightarrow \mathbb{C}$ is *satisfactory on* $[0, k)$ if (and only if) there exist coefficients b_n and a number $k > 0$ such that

$$\varphi(\sigma) := \sum_{n=0}^{\infty} b_n \sigma^{2n+1} \quad \text{for} \quad 0 \leq \sigma < k.$$

Lemma 3.4. (i) The function βg is satisfactory on $[0, (2\pi)^{1/2})$.

(ii) If f is satisfactory on $[0, (2\pi)^{1/2})$, then so is $\mathcal{G}f$.

Proof. (i) We note that

$$\beta(\sigma)g(\sigma) = \sigma e^{-\sigma^2} \left(\frac{3}{2} + \frac{\sigma^2}{1 - e^{-\sigma^2}} \left(\int_{\sigma}^1 \frac{1 - e^{-\rho^2}}{\rho} d\rho + C_0 + C_1 \right) \right),$$

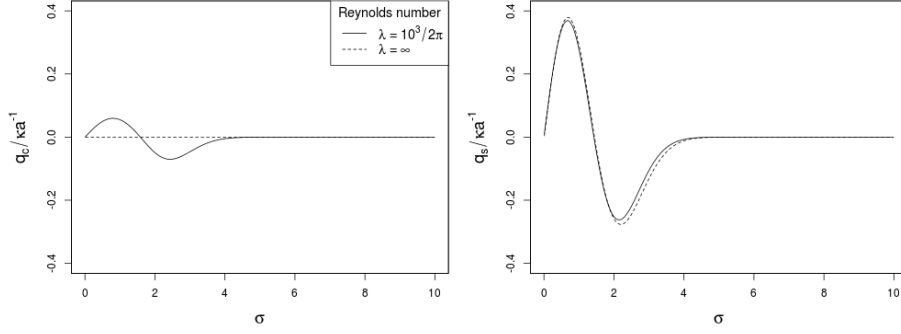


Figure 2: The perturbations for $q_c(\alpha)$ and $q_s(\alpha)$ for $\lambda = 10^3/2\pi$ and $\lambda = \infty$.

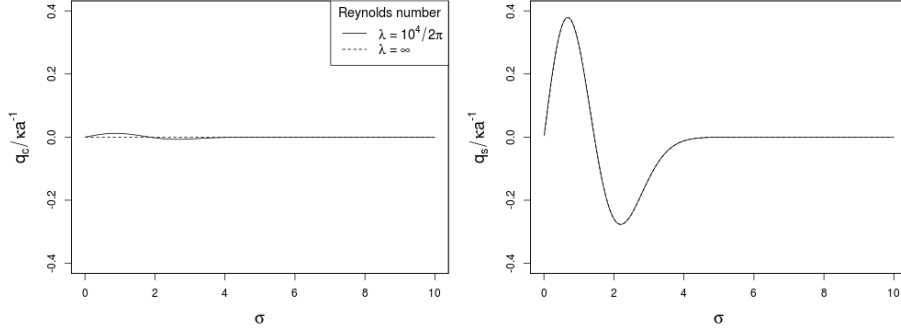


Figure 3: The perturbations for $q_c(\alpha)$ and $q_s(\alpha)$ for $\lambda = 10^4/2\pi$ and $\lambda = \infty$.

where

$$C_0 := - \int_1^\infty \frac{e^{-\rho^2}}{\rho} d\rho, \quad C_1 := -\frac{1}{2} (\gamma_E + 1 - \log 2),$$

and that the function w defined by

$$w(z) = \frac{z}{1 - e^{-z}} \quad \text{if } z \in \mathbb{C} \setminus \{0\} \setminus \{\text{poles}\} \quad \text{and} \quad w(0) = 1,$$

is holomorphic for $|z| < 2\pi$.

(ii) Let

$$W(\sigma) := \frac{1}{\sigma} - U(\sigma) \log \frac{e^{\sigma^2} - 1}{\sigma^2},$$

where the limiting value of $(e^{\sigma^2} - 1)/\sigma^2$ is taken at $\sigma = 0$. Then

$$V(\sigma) = W(\sigma) - 2U(\sigma) \log \sigma$$

and

$$(\mathcal{G}f)(\sigma) = \frac{1}{2}W(\sigma) \int_0^\sigma U(\rho)f(\rho)\rho d\rho - \frac{1}{2}U(\sigma) \int_0^\sigma W(\rho)f(\rho)\rho d\rho \\ + \sigma^2 U(\sigma) \int_0^1 (\log t)U(\sigma t)f(\sigma t)tdt.$$

The functions with values $\sigma^2 W(\sigma)$, $U(\sigma)$ and $f(\sigma)$ are all satisfactory on $[0, (2\pi)^{1/2})$, so that $\mathcal{G}f$ inherits this property. \square

Theorem 3.5. *The perturbation $\tilde{\omega}_1$ has a representation*

$$\tilde{\omega}_1(\sigma, \theta; \lambda) = \cos \theta \{ \lambda^{-1}\zeta_1(\sigma) + \lambda^{-3}\zeta_3(\sigma) + \zeta_5(\sigma; \lambda) \} \\ + \sin \theta \{ \zeta_0(\sigma) + \lambda^{-2}\zeta_2(\sigma) + \zeta_4(\sigma; \lambda) \} \quad (3.25)$$

in which, for $m = 0, 1, 2, 3$ and $n = 4, 5$,

(a) the functions ζ_m and $\zeta_n(\cdot; \lambda)$ belong to $C^\infty[0, \infty)$ and are satisfactory on $[0, (2\pi)^{1/2})$;

(b) as $\sigma \uparrow \infty$, $\zeta_m(\sigma) = O(\sigma^{2m+4}e^{-\sigma^2})$ and $\zeta_n(\sigma; \lambda) = o(e^{-\sigma^2/2})$ for fixed λ ;

(c) as $\lambda \uparrow \infty$, $\zeta_n(\sigma; \lambda) = O(\lambda^{-n})$, uniformly over $\sigma \in [0, \infty)$.

The proof will be by means of further lemmas. Let $\Psi := \tilde{\psi}_{1c} + i\tilde{\psi}_{1s}$, as before, and let $\Omega := -\Delta_1\Psi$, so that $\Omega = \tilde{\omega}_{1c} + i\tilde{\omega}_{1s}$. Our plan is to construct identities

$$\Psi(\sigma; \lambda) = \sum_{n=0}^N \lambda^{-n}\Psi_n(\sigma) + R_N(\sigma; \lambda), \quad (3.26)$$

$$\Omega(\sigma; \lambda) = \sum_{n=0}^N \lambda^{-n}\Omega_n(\sigma) + r_N(\sigma; \lambda), \quad (3.27)$$

in which estimates of the remainders R_N and r_N can be crude. In fact, we shall prove only that R_N and r_N are $O(\lambda^{1-N})$, but this is sufficient for (3.25) if $N \geq 6$.

The terms of the expansion of Ψ are to satisfy

$$\{\Delta_1 + \alpha(\sigma)\}\Psi_n = h_n, \quad n = 0, 1, \dots, N, \quad (3.28)$$

where

$$h_0(\sigma) = -i\frac{\kappa}{\pi a}\beta(\sigma)g(\sigma), \quad (3.29)$$

$$h_n := i\beta\mathcal{E}(\Delta_1\Psi_{n-1}) \quad \text{for } n = 1, \dots, N, \quad (3.30)$$

and

$$-i\lambda^{-1}\beta(\sigma)\mathcal{E}(\Delta_1 R_N) + \{\Delta_1 + \alpha(\sigma)\}R_N = i\lambda^{-N-1}\beta(\sigma)\mathcal{E}(\Delta_1\Psi_N); \quad (3.31)$$

then the right-hand member of (3.26) will satisfy the equation (3.5) governing Ψ .

Since $\Omega = -\Delta_1\Psi$ and $\Psi = T_1\Omega$, this scheme corresponds to

$$-\Omega_n + \alpha(\sigma)T_1\Omega_n = h_n, \quad n = 0, 1, \dots, N, \quad (3.32)$$

$$i\lambda^{-1}\beta(\sigma)\mathcal{E}r_N - r_N + \alpha(\sigma)T_1r_N = -i\lambda^{-N-1}\beta(\sigma)\mathcal{E}\Omega_N, \quad (3.33)$$

where $h_n = -i\beta(\sigma)\mathcal{E}\Omega_{n-1}$ for $n = 1, \dots, N$.

Lemma 3.6. *In order that equation (3.28), with the side conditions (3.6), have a solution, it is necessary that*

$$\int_0^\infty U(\sigma)h_n(\sigma)\sigma d\sigma = 0, \quad n = 0, 1, \dots, N; \quad (3.34)$$

equivalently, that

$$\int_0^\infty \sigma^2 g(\sigma) d\sigma = 0 \quad \text{if } n = 0, \quad (3.35)$$

$$\int_0^\infty \sigma^2 (\mathcal{E}\Omega_{n-1})(\sigma) d\sigma = 0 \quad \text{if } n = 1, \dots, N. \quad (3.36)$$

Proof. Let $\mathcal{M} := \Delta_1 + \alpha(\sigma)$. Assume that u and v are in $C^2(0, \infty)$, that $\sigma u(\sigma)v'(\sigma) \rightarrow 0$ as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$ and that $\sigma u'(\sigma)v(\sigma) \rightarrow 0$ as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$. Then integration by parts yields

$$\int_0^\infty u(\mathcal{M}v)\sigma d\sigma = \int_0^\infty (\mathcal{M}u)v\sigma d\sigma. \quad (3.37)$$

Now let $u = U$ and $v = \Psi_n$. If Ψ_n satisfies (3.6), then the foregoing hypotheses are satisfied. If also $\mathcal{M}\Psi_n = h_n$, then

$$\int_0^\infty U h_n \sigma d\sigma = \int_0^\infty U(\mathcal{M}\Psi_n)\sigma d\sigma = \int_0^\infty (\mathcal{M}U)\Psi_n \sigma d\sigma = 0. \quad (3.38)$$

Equations (3.35) and (3.36) follow from the identity $U(\sigma)\beta(\sigma) = \sigma$ and from the definitions of h_n . \square

If h_n satisfies not only the orthogonality condition (3.34), but also the other hypotheses on f in Lemma 3.3 (and this will be the case), then the differential equation (3.28), with side conditions (3.6), has solutions

$$\Psi_n = c_n U - \mathcal{G}h_n, \quad n = 0, 1, \dots, N, \quad (3.39)$$

whence

$$\Omega_n(\sigma) = -(\Delta_1 \Psi_n)(\sigma) = 4c_n \sigma e^{-\sigma^2} - \alpha(\sigma)(\mathcal{G}h_n)(\sigma) - h_n(\sigma), \quad (3.40)$$

for every $c_n \in \mathbb{C}$.

In order to evaluate c_0, \dots, c_N and in order to discuss r_N , we extend the definition (3.30) to h_{N+1} . Recall from Lemma 3.2 that for $n = 0$ the orthogonality condition (3.34) has already been established.

Lemma 3.7. *For $n = 0, 1, \dots, N$, the necessary condition $\int_0^\infty U h_{n+1} \sigma d\sigma = 0$ implies that $c_n = -\frac{1}{2}J(h_n)$, where $J(\cdot)$ is defined by (3.15).*

Proof. Extended to Ω_N , the orthogonality condition (3.36) states that, for $n = 0, 1, \dots, N$,

$$0 = \int_0^\infty \sigma^2 (\mathcal{E}\Omega_n) d\sigma = -4 \int_0^\infty \sigma^2 \Omega_n d\sigma,$$

by an integration by parts for which it suffices that $\Omega_n \in C^2(0, \infty)$, that $\Omega'_n(\sigma) = o(\sigma^{-2})$ both as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$, that $\Omega_n = o(\sigma^{-1})$ as $\sigma \downarrow 0$ and that $\Omega_n = o(\sigma^{-3})$ as $\sigma \uparrow \infty$.

Next, we observe that, if $\Psi_n \in C^2(0, \infty)$ and if both $\sigma^2 \Psi'_n(\sigma)$ and $\sigma \Psi_n(\sigma)$ have limits both as $\sigma \downarrow 0$ and as $\sigma \uparrow \infty$, then

$$0 = \int_0^\infty \sigma^2 (\Delta_1 \Psi_n) d\sigma = [\sigma^2 \Psi'_n - \sigma \Psi_n]_0^\infty,$$

in which limiting values are implied on the right-hand side. In view of (3.39), the orthogonality condition is now

$$c_n [\sigma^2 U' - \sigma U]_0^\infty = [\sigma^2 (\mathcal{G}h_n)' - \sigma (\mathcal{G}h_n)]_0^\infty.$$

Referring to the definition of U in (3.9) and to the description of $\mathcal{G}f$ in Lemma 3.3, one is led to $c_n = -\frac{1}{2}J(h_n)$. \square

It is time to relate the ζ_m and $\zeta_n(\cdot; \lambda)$ in Theorem 3.5 to the Ω_n and r_N in (3.27). We noted after (3.23) that Q_0 is imaginary. Since $\Omega(\sigma) = \exp(-\sigma^2/2)Q(\sigma)$, the function Ω_0 is imaginary. Then, since $h_1 = -i\beta\mathcal{E}\Omega_0$, the function h_1 and the coefficient c_1 are real. Equation (3.40) shows that Ω_1 is real. An easy induction now shows that Ω_n is imaginary if n is even and Ω_n is real if n is odd.

Accordingly, if N is odd, then

$$\begin{aligned} \zeta_1 = \Omega_1, \quad \zeta_3 = \Omega_3 \text{ and } \zeta_5(\cdot; \lambda) &= \lambda^{-5}\Omega_5 + \dots + \lambda^{-N}\Omega_N + \operatorname{Re} r_N(\cdot; \lambda), \\ \zeta_0 = -i\Omega_0, \quad \zeta_2 = -i\Omega_2 \text{ and } \zeta_4(\cdot; \lambda) &= -i(\lambda^{-4}\Omega_4 + \dots + \lambda^{-N+1}\Omega_{N-1}) \\ &\quad + \operatorname{Im} r_N(\cdot; \lambda). \end{aligned} \quad (3.41)$$

If N is even, then there is a similar array.

Because of the explicit formula (3.40) for Ω_n (with $h_n = -i\beta\mathcal{E}\Omega_{n-1}$, with $c_n = -\frac{1}{2}J(h_n)$ and with the operator \mathcal{G} described by Lemmas 3.3 and 3.4), enough may have been said about Ω_n to justify the claims made for ζ_0 to ζ_3 in Theorem 3.5. For example, the result

$$\zeta_m(\sigma) = O(\sigma^{2m+4}e^{-\sigma^2}) \quad \text{as } \sigma \uparrow \infty \quad (3.42)$$

follows for $m = 0$ from $\beta(\sigma) \sim \sigma^2$ and from the overestimate $g(\sigma) = O(\sigma^2 e^{-\sigma^2})$, which imply that h_0 and Ω_0 are $O(\sigma^4 e^{-\sigma^2})$. Then repeated use of $h_{n+1} = -i\beta\mathcal{E}\Omega_n$ leads to (3.42).

On the other hand, the remainder r_N requires further discussion. Under the transformations

$$r_N(\sigma) = e^{-\sigma^2/2}P_N(\sigma) = e^{-\sigma^2/2} \{p_{Nc}(\sigma) + ip_{Ns}(\sigma)\}, \quad (3.43)$$

$$p_N(\sigma, \theta) := p_{Nc}(\sigma) \cos \theta + p_{Ns}(\sigma) \sin \theta, \quad (3.44)$$

equation (3.33) becomes

$$-(\Delta_\sigma - \sigma^2)p_N + \frac{\lambda}{\beta(\sigma)} \frac{\partial}{\partial \theta} p_N - 4\lambda e^{-\sigma^2/2} T(e^{-\sigma^2/2} p_N) = \lambda^{-N} e^{\sigma^2/2} f_N(\sigma, \theta), \quad (3.45)$$

where

$$f_N(\sigma, \theta) = \operatorname{Re} \{(\mathcal{E}\Omega_N)(\sigma) e^{-i\theta}\}. \quad (3.46)$$

The operator on the left-hand side of (3.45) is that in (2.6); as in §2, it follows that equation (3.45) has a unique weak solution bounded by

$$\|p_N\|_Z \leq \frac{\kappa}{a} A_N \lambda^{-N}, \quad (3.47)$$

where A_N depends only on N .

Choosing the test function in the definition of weak solution as in (2.18), we obtain the equation

$$\begin{aligned} P_N(\rho) &= \lambda^{-N} \int_0^\infty K(\rho, \sigma) e^{\sigma^2/2} (\mathcal{E}\Omega_N)(\sigma) \sigma d\sigma \\ &\quad + i\lambda \int_0^\infty K(\rho, \sigma) \left(\frac{P_N(\sigma)}{\beta(\sigma)} - 4e^{-\sigma^2/2} T_1(e^{-\sigma^2/2} P_N) \right) \sigma d\sigma. \end{aligned} \quad (3.48)$$

This leads without difficulty to a pointwise solution $P_N \in C^2[0, \infty)$ such that $P_N(0) = 0$ and such that $P_N(\sigma) \rightarrow 0$ as $\sigma \uparrow \infty$. (Correspondingly, $r_N(\sigma)$ is $o(\exp(-\sigma^2/2))$ as $\sigma \uparrow \infty$.) Equation (3.48) and the bound (3.47) now imply that $P_N(\sigma; \lambda)$ is $O(\lambda^{-N+1})$ uniformly over $\sigma \in [0, \infty)$. Moreover, the equation

$$-(\Delta_1 - \sigma^2)P_N - \frac{i\lambda}{\beta(\sigma)}P_N + 4i\lambda e^{-\sigma^2/2} T_1(e^{-\sigma^2/2} P_N) = \lambda^{-N} e^{\sigma^2/2} \mathcal{E}\Omega_N,$$

may be written as

$$P_N'' = -\frac{1}{\sigma} P_N' + \left(\frac{1}{\sigma^2} + \sigma^2 \right) P_N - \dots - \lambda^{-N} e^{\sigma^2/2} \mathcal{E}\Omega_N.$$

The right-hand member of this is in $C^1[k, \infty)$ for any $k > 0$, say in $C^1[1, \infty)$. Therefore $P_N'' \in C^1[1, \infty)$. Repetition of this step shows that $P_N \in C^\infty[1, \infty)$.

It remains to prove that P_N is better than C^2 at and near the origin. We return to equation (3.33) for r_N and to the equation

$$\mathcal{E}\Omega + \frac{i\lambda}{\beta(\sigma)}\Omega - 4i\lambda e^{-\sigma^2} T_1\Omega = -\frac{\kappa\lambda}{\pi a}g$$

for $\Omega = \tilde{\omega}_{1c} + i\tilde{\omega}_{1s}$; our final lemma applies to both r_N and Ω .

Lemma 3.8. *Assume that the equation*

$$\mathcal{E}u + \frac{i\lambda}{\beta(\sigma)}u - 4i\lambda e^{-\sigma^2} T_1u = \lambda f \quad (3.49)$$

has a solution $u \in C^2[0, \infty)$ such that $u(0) = 0$ and such that $u(\sigma)$ is $o(\exp(-\sigma^2/2))$ as $\sigma \uparrow \infty$. Assume also that u is unique because it is the transformed version of a solution in the Hilbert space Z .

Then u is satisfactory on $[0, (2\pi)^{1/2})$ whenever f has this property.

Proof. We shall prove that there are coefficients a_n such that

$$u(\sigma) = \sum_{n=0}^{\infty} a_n \sigma^{2n+1} \quad \text{for } 0 \leq \sigma < b$$

if $b \in (0, (2\pi)^{1/2})$. Here we are not constructing a series solution *ab initio* in the usual way; rather, we are establishing a regularity property of a known,

unique solution. Therefore we may regard $a_0 = u'(0)$ and $\int_0^\infty u d\sigma$ as known; we proceed to calculate the other coefficients in terms of these. The equation is satisfied, subject to convergence of the series, if for $n = 0, 1, 2, \dots$

$$4(n+1)(n+2)a_{n+1} = -4(n+1)a_n - \sum_{j=0}^n (B_{n-j}a_j + A_{n-j}\tau_j) + \lambda f_n, \quad (3.50)$$

where

$$B_m = \frac{i\lambda(-1)^m}{(m+1)!}, \quad A_m = \frac{4i\lambda(-1)^{m+1}}{m!},$$

$$\tau_0 = \frac{1}{2} \int_0^\infty u d\sigma, \quad \tau_m = -\frac{1}{4} \frac{a_{m-1}}{m(m+1)} \text{ for } m \geq 1,$$

and

$$f(\sigma) = \sum_{n=0}^{\infty} f_n \sigma^{2n+1} \quad \text{for } 0 \leq \sigma < (2\pi)^{1/2}.$$

Hence there is a constant $C = C(b)$ such that $|f_n| \leq Cb^{-2n}$.

Now, for every $p \in \{1, 2, 3, \dots\}$ there is a number $\Gamma_p = \Gamma_p(b, \lambda)$ such that $|a_n| \leq \Gamma_p b^{-2n}$ for $n = 0, 1, \dots, p$. We may suppose that $\Gamma_p \geq \kappa/\alpha$. Then (3.50) implies that

$$|a_{n+1}| \leq \Gamma_p b^{-2n-2} \varphi(n, \lambda) \quad \text{for } n \leq p,$$

where

$$\varphi(n, \lambda) := \frac{2\pi}{4(n+1)(n+2)} \left(4(n+1) + \lambda(1+\pi)e^{2\pi} + \frac{4\lambda|\tau_0|(2\pi)^n}{n!} + \lambda C \right).$$

For fixed λ , we choose p so large that $\varphi(p, \lambda) \leq 1$ and so large that $\varphi(n, \lambda)$ decreases for $n \geq p$. Then $|a_m| \leq \Gamma_p b^{-2m}$ not merely for $m \leq p$, but also for $m \geq p+1$. \square

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